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VARIATIONAL FORMULATIONS FOR NON-RELATIVISTIC AND RELATIVISTIC FLUID MOTIONS IN AN EULERIAN CONTEXT

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Abstract

In its first part, this article develops a variational formulation for a fluid motion in a non-relativistic context. The results are obtained through standard tools of calculus of variations and constrained optimization in function spaces. In order to include a scalar field of temperature as a variable, concerning the main functional, we assume a rather standard energy equation as a constraint. In the last section, we also address a fluid motion in a relativistic context. Finally, we emphasize the modeling approach here addressed is essentially an Eulerian one.

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1 Introduction

In its first part, this article we develops a variational formulation for a fluid motion in a non-relativistic context.

We assume a total mass restriction as a constraint along a period of time denoted by $[0, t_f]$.

The results are obtained through standard tools of calculus of variations, basic constrained optimization and related Lagrange multipliers approach in infinite dimensional spaces.

As standard references in theoretical fluid mechanics we would cite [1, 2, 3, 4, 5, 6]. For related numerical methods we would mention [7, 8, 9]. Concerning similar models, we would cite [10, 11, 12, 13, 14].

Finally, details on the Sobolev spaces involved may be found in [15].

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Consider a fluid motion initially modeled as a particle system one, through the field of position $\mathbf{r} : \Omega \times [0, t_f] \to \mathbb{R}$ and field of velocities $\mathbf{v}$, where $\mathbf{v}(\mathbf{r}(x,t),t)$ stands for the velocity at the position $\mathbf{r}(x,t)$ and time $t \in [0, t_f]$.

2 A variational formulation for a compressible fluid motion

In this section, we develop in details the concerning variational formulation for a compressible case.

Assuming the Einstein convection of summing up repeated indices and denoting

$$\mathbf{r}(x,t) = (X_1(x,t), X_2(x,t), X_3(x,t)),$$

for such a compressible motion we also assume the mass conservation equation as a constraint, which stands for

$$\frac{d\rho}{dt} + \rho \frac{\partial}{\partial x_j}\left(\frac{\partial X_j(x,t)}{\partial t}\right) = 0, \text{ in } \Omega \times [0, t_f].$$

Moreover, we suppose a non-constant field of temperature $T(\mathbf{r}(x,t),t)$ and a non-constant fluid density $\rho = \rho(\mathbf{r}(x,t),t) > 0$ so that the macroscopic system kinetics energy functional stands for

$$E_c = \frac{1}{2}\int_0^T \int_\Omega \rho \mathbf{v}(\mathbf{r}(x,t),t) \cdot \mathbf{v}(\mathbf{r}(x,t),t)\, dx\, dt.$$

For such a modeling, denoting by $\mathbf{g} = -g\mathbf{k}$ the gravity field, we assume such a system is a particle one, comprised by a number N of particles, with position fields

$$\mathbf{r}_j(x,t) = \mathbf{r}(x,t) + \hat{\mathbf{r}}_j(\mathbf{r}(x,t),t),\ \forall j \in \{1,\cdots,N\}.$$

Here $\hat{\mathbf{r}}_j(\mathbf{r}(x,t),t)$ refers to internal vibrational degrees of freedom related to the internal energy and temperature concepts.

We assume the following constraints,

$$C_v T = \sum_{j=1}^{N} \frac{1}{2}|\phi_j|^2 \frac{d\hat{\mathbf{r}}_j(\mathbf{r}(x,t),t)}{dt} \cdot \frac{d\hat{\mathbf{r}}_j(\mathbf{r}(x,t),t)}{dt},$$

in $\Omega \times [0,t_f]$, where

$$|\phi_j(\mathbf{r}(x,t),t)|^2$$

is the scalar density field of the particle p_j, and following the approach in reference [16],

$$C_v\rho\frac{dT(\mathbf{r}(x,t),t)}{dt} - K\nabla^2 T - \rho\mathbf{g}\cdot\mathbf{v} + P\ \text{div}\ (\mathbf{v}) - \frac{\partial Q}{\partial t} = 0,$$

in $\Omega \times [0,t_f]$, where as previously indicated,

$$T = T(\mathbf{r}(x,t),t)$$

is the temperature scalar field, $C_v > 0$, $K > 0$ are real constants and Q is a heat function.

We define also the mass restrictions,

$$\int_\Omega |\phi_j(\mathbf{r}(x,t),t)|^2\, dx = m_{p_j},\ \text{in}\ [0,t_f],$$

$$\rho(\mathbf{r}(x,t),t) = \sum_{j=1}^{N} |\phi_j(\mathbf{r}(x,t),t)|^2,\ \text{in}\ \Omega \times [0,t_f],$$

where m_{p_j} is the mass of a particle p_j.

As usual, we generically denote

$$\langle f,h\rangle_{L^2} = \int_0^{t_f}\int_\Omega f\ h\ dxdt,\ \forall f,h \in L^2(\Omega\times[0,t_f]),$$

with an analogous notation for a vectorial case.

Concerning such a fluid motion modeling, denoting generically $Y = C^1$, $Y_1 = C^2$ and $V = C_0^2$, we define the following variational formulation represented by

the functional $J : Y \times V \times Y_1 \to \mathbb{R}$ already including the concerning Lagrange multipliers for the constraints, where

$$
\begin{aligned}
& J(\mathbf{v}, \mathbf{r}, \hat{\mathbf{r}}, \phi, \rho, T, \lambda, \hat{P}, E) \\
= \; & \frac{1}{2} \int_0^{t_f} \int_\Omega \rho \mathbf{v}(\mathbf{r}(x,t),t) \cdot \mathbf{v}(\mathbf{r}(x,t),t) \; dx \; dt \\
& + \frac{1}{2} \sum_{j=1}^{N} \int_0^{t_f} \int_\Omega |\phi_j(\mathbf{r}(x,t),t)|^2 \frac{d\hat{\mathbf{r}}_j(\mathbf{r}(x,t),t)}{dt} \cdot \frac{d\hat{\mathbf{r}}_j(\mathbf{r}(x,t),t)}{dt} \; dxdt \\
& + \sum_{j=1}^{N} \int_0^{t_f} \int_\Omega |\phi_j(\mathbf{r}(x,t),t)|^2 \frac{d\hat{\mathbf{r}}_j(\mathbf{r}(x,t),t)}{dt} \cdot \frac{d\mathbf{r}(x,t)}{dt} \; dxdt \\
& + \left\langle \lambda, \mathbf{v}(\mathbf{r}(x,t),t) - \frac{\partial \mathbf{r}(x,t)}{\partial t} \right\rangle_{L^2} \\
& - \left\langle \hat{P}, \frac{d\rho}{dt} + \rho \frac{\partial}{\partial x_j} \left(\frac{\partial X_j(x,t)}{\partial t} \right) \right\rangle_{L^2} \\
& - \int_0^{t_f} \int_\Omega \rho \mathbf{g} \cdot \mathbf{r}(x,t) \; dxdt \\
& + \int_0^{t_f} \int_\Omega \lambda_1 \left(\rho \, C_v \, \frac{d\,T}{dt} - K \nabla^2 T + P \; \mathrm{div} \; \mathbf{v} - \rho \mathbf{g} \cdot \mathbf{v} \right) \; dxdt \\
& + \int_0^{t_f} \int_\Omega \lambda_2 \left(C_v \, T - \frac{1}{2} \sum_{j=1}^{N} |\phi_j|^2 \frac{d\hat{\mathbf{r}}_j(\mathbf{r}(x,t),t)}{dt} \cdot \frac{d\hat{\mathbf{r}}_j(\mathbf{r}(x,t),t)}{dt} \right) \; dxdt \\
& + \frac{1}{2} \sum_{j=1}^{N} \gamma_j \int_0^{t_f} \int_\Omega \nabla \phi_j \cdot \nabla \phi_j \; dxdt \\
& + \frac{1}{2} \sum_{j=1}^{N} \gamma_0 \int_0^{t_f} \int_\Omega \frac{d\phi_j}{dt} \frac{d\phi_j}{dt} \; dxdt \\
& + \sum_{j=1}^{N} \sum_{k=1}^{N} \int_0^{t_f} \int_\Omega \int_\Omega \alpha_{jk} \frac{|\phi_j(\mathbf{r}(x,t),t)|^2 |\phi_j(\mathbf{r}(\tilde{x},t),t)|^2}{|\mathbf{r}(x,t) - \mathbf{r}(\tilde{x},t)|} \; dx \; d\tilde{x} \; dt \\
& - \sum_{j=1}^{N} \int_0^{t_f} E_j(t) \left(\int_\Omega |\phi_j(\mathbf{r}(x,t),t)|^2 \; dx - m_{p_j} \right) \; dt \\
& + \int_0^{t_f} \int_\Omega \lambda_3 \left(\rho(\mathbf{r}(x,t),t) - \sum_{j=1}^{N} |\phi_j(\mathbf{r}(x,t),t)|^2 \right) \; dxdt. \qquad (1)
\end{aligned}
$$

where the pressure field P is defined by

$$P = P(\hat{P}, \rho) = \frac{d(\hat{P}\rho)}{dt},$$

$$\gamma_j > 0, \ \forall j \in \{1, \cdots, N\}$$

and $\gamma_0 > 0$ are appropriate real constants, and

$$\{\alpha_{jk}\}_{N \times N}$$

is an appropriate positive definite real matrix.

3 The passage from a discrete model to a continuous one

Observe that in general N is very large.

With such an issue in mind, we shall propose to replace the discrete variables

$$\{\hat{\mathbf{r}}_j(\mathbf{r}(x,t),t), \ j \in \{1, \cdots, N\}\},$$

by a continuous one in s

$$\hat{\mathbf{r}}(\mathbf{r}(x,t),t,s)$$

where $s \in [0,b]$, for some appropriate large real constant $b > 0$.

Similarly, we shall replace the discrete fields

$$\{\phi_j(\mathbf{r}(x,t),t), \ j \in \{1, \cdots, N\}\},$$

by a continuous one in s

$$\phi(\mathbf{r}(x,t),t,s)$$

and the real constants

$$\{\gamma_j, \alpha_{jk}, \ j,k \in \{1, \cdots N\}\}$$

by appropriate positive functions

$$\gamma(s) \text{ and } \alpha(s,p),$$

where again $s, p \in [0,b]$.

Thus, with such replacements we redefine the functional J, now denoting it by J_1, as

$$
\begin{aligned}
& J_1(\mathbf{v},\mathbf{r},\hat{\mathbf{r}},\phi,\rho,T,\lambda,\hat{P},E,E_1,N) \\
= \; & \frac{1}{2}\int_0^{t_f}\int_\Omega \rho\mathbf{v}(\mathbf{r}(x,t),t)\cdot\mathbf{v}(\mathbf{r}(x,t),t)\,dx\,dt \\
& +\frac{1}{2}\int_0^b\int_0^{t_f}\int_\Omega |\phi(\mathbf{r}(x,t),t,s)|^2\frac{d\hat{\mathbf{r}}(\mathbf{r}(x,t),t,s)}{dt}\cdot\frac{d\hat{\mathbf{r}}(\mathbf{r}(x,t),t,s)}{dt}\,dxdtds \\
& +\int_0^b\int_0^{t_f}\int_\Omega |\phi(\mathbf{r}(x,t),t,s)|^2\frac{d\hat{\mathbf{r}}(\mathbf{r}(x,t),t,s)}{dt}\cdot\frac{d\mathbf{r}(x,t)}{dt}\,dxdtds \\
& +\left\langle \lambda, \mathbf{v}(\mathbf{r}(x,t),t)-\frac{\partial\mathbf{r}(x,t)}{\partial t}\right\rangle_{L^2} \\
& -\left\langle \hat{P}, \frac{d\rho}{dt}+\rho\frac{\partial}{\partial x_j}\left(\frac{\partial X_j(x,t)}{\partial t}\right)\right\rangle_{L^2} \\
& -\int_0^{t_f}\int_\Omega \rho\mathbf{g}\cdot\mathbf{r}(x,t)\,dxdt \\
& +\int_0^{t_f}\int_\Omega \lambda_1\left(\rho C_v\frac{dT}{dt}-K\nabla^2T+P(\hat{P},\rho)\ \mathrm{div}\ \mathbf{v}-\rho\mathbf{g}\cdot\mathbf{v}\right)\,dxdt \\
& +\int_0^{t_f}\int_\Omega \lambda_2\,(C_v\,T \\
& -\frac{1}{2}\int_0^b |\phi(\mathbf{r}(x,t),t,s)|^2\frac{d\hat{\mathbf{r}}(\mathbf{r}(x,t),t,s)}{dt}\cdot\frac{d\hat{\mathbf{r}}(\mathbf{r}(x,t),t,s)}{dt}\,ds\Big)\,dxdt \\
& +\frac{1}{2}\int_0^b\gamma(s)\int_0^{t_f}\int_\Omega \nabla\phi(\mathbf{r}(x,t),t,s)\cdot\nabla\phi(\mathbf{r}(x,t),t,s)\,dsdxdt \\
& +\frac{1}{2}\int_0^b\gamma_0\int_0^{t_f}\int_\Omega \frac{d\phi(\mathbf{r}(x,t),t,s)}{dt}\frac{d\phi(\mathbf{r}(x,t),t,s)}{dt}\,ds\,dxdt \\
& +\int_0^b\int_0^b\int_0^{t_f}\int_\Omega\int_\Omega \alpha(s,p)\frac{|\phi(\mathbf{r}(x,t),t,s)|^2|\phi(\mathbf{r}(\tilde{x},t),t,p)|^2}{|\mathbf{r}(x,t)-\mathbf{r}(\tilde{x},t)|}\,dx\,d\tilde{x}\,dt\,dsdp \\
& -\int_0^b\int_0^{t_f} E(t,s)\left(\int_\Omega |\phi(\mathbf{r}(x,t,s),t)|^2\,dx-N(s,t)\right)\,dt\,ds \\
& +\int_0^{t_f} E_1(t)\left(\int_0^b N(s,t)\,ds-m_T\right)\,dt \\
& +\int_0^{t_f}\int_\Omega \lambda_3\left(\rho(\mathbf{r}(x,t),t)-\int_0^b |\phi(\mathbf{r}(x,t),t,s)|^2\,ds\right)\,dxdt, \qquad (2)
\end{aligned}
$$

where we recall the pressure field is defined by

$$P = P(\hat{P}, \rho) = \frac{d(\rho \hat{P})}{dt}$$

and m_T is the fixed total system mass.

Remark 3.1. *We consider variations in the spaces $C_c^\infty(\Omega \times [0, t_f])$ and $C_c^\infty(\Omega \times [0, t_f] \times [0, b])$ and their vectorial analogous ones, so that we obtain the related Euler-lagrange equations in a distributional sense on these domains $\Omega \times [0, t_f]$ and $\Omega \times [0, t_f] \times [0, b]$ not addressing the concerning boundary and initial conditions in this first work.*

Symbolically, the variation of J in $\mathbf{v}$ stands for

$$\rho \mathbf{v} + \lambda - \nabla(\lambda_1 P) - \lambda_1 \rho \mathbf{g} = \mathbf{0},$$

in $\Omega \times [0, t_f]$.

Moreover, the variation of J in $\mathbf{r}$ stands for

$$\begin{aligned}
\mathbf{0} = & \frac{dJ_1(\mathbf{v}, \mathbf{r}, \hat{\mathbf{r}}, \phi, \rho, T, \lambda, \hat{P}, E, E_1, N)}{d\mathbf{r}} \\
= & \frac{\partial J_1(\mathbf{v}, \mathbf{r}, \hat{\mathbf{r}}, \phi, \rho, T, \lambda, \hat{P}, E, E_1, N)}{\partial \mathbf{v}} \frac{\partial \mathbf{v}}{\partial \mathbf{r}} \\
& + \frac{\partial J_1(\mathbf{v}, \mathbf{r}, \hat{\mathbf{r}}, \phi, \rho, T, \lambda, \hat{P}, E, E_1, N)}{\partial \rho} \frac{\partial \rho}{\partial \mathbf{r}} \\
& + \frac{\partial J_1(\mathbf{v}, \mathbf{r}, \hat{\mathbf{r}}, \phi, \rho, T, \lambda, \hat{P}, E, E_1, N)}{\partial \hat{\mathbf{r}}} \frac{\partial \hat{\mathbf{r}}}{\partial \mathbf{r}} \\
& + \frac{\partial J_1(\mathbf{v}, \mathbf{r}, \hat{\mathbf{r}}, \phi, \rho, T, \lambda, \hat{P}, E, E_1, N)}{\partial \phi} \frac{\partial \phi}{\partial \mathbf{r}} \\
& + \frac{\partial J_1(\mathbf{v}, \mathbf{r}, \hat{\mathbf{r}}, \phi, \rho, T, \lambda, \hat{P}, E, E_1, N)}{\partial T} \frac{\partial T}{\partial \mathbf{r}} \\
& + \frac{\partial J_1(\mathbf{v}, \mathbf{r}, \hat{\mathbf{r}}, \phi, \rho, T, \lambda, \hat{P}, E, E_1, N)}{\partial \mathbf{r}} \\
= & \frac{\partial J_1(\mathbf{v}, \mathbf{r}, \hat{\mathbf{r}}, \phi, \rho, T, \lambda, \hat{P}, E, E_1, N)}{\partial \mathbf{r}}.
\end{aligned} \tag{3}$$

In summary, we must have

$$\frac{\partial J_1(\mathbf{v}, \mathbf{r}, \hat{\mathbf{r}}, \phi, \rho, T, \lambda, \hat{P}, E, E_1, N)}{\partial \mathbf{r}} = \mathbf{0},$$

so that

$$\frac{\partial \lambda}{\partial t} - \frac{d}{dt}\left(\int_0^b |\phi(\mathbf{r}(x,t),t,s)|^2 \frac{d\hat{\mathbf{r}}(\mathbf{r}(x,t),t,s)}{dt}\, ds\right)$$
$$-\int_\Omega \int_0^b \int_0^b \frac{|\phi(\mathbf{r}(x,t),t,s)|^2 |\phi(\mathbf{r}(\tilde{x},t),t,p)|^2 (\mathbf{r}(x,t)-\mathbf{r}(\tilde{x},t))}{|\mathbf{r}(x,t)-\mathbf{r}(\tilde{x},t)|^3}\, d\tilde{x}dsdp$$
$$-\rho \mathbf{g} = \mathbf{0}, \tag{4}$$

in $\Omega \times [0, t_f]$.

Denoting

$$\hat{v} = \frac{d\hat{\mathbf{r}}(\mathbf{r}(x,t),t,s)}{dt}$$

and combining these last two variations, we obtain

$$\frac{d(\rho \mathbf{v})}{dt} + \frac{d\nabla(\rho \hat{P})}{dt} - \frac{d(\nabla(\lambda_1 P))}{dt} - \frac{d(\lambda_1 \rho \mathbf{g})}{dt}$$
$$+\frac{d}{dt}\left(\int_0^b |\phi(\mathbf{r}(x,t),t,s)|^2 \hat{v}\, ds\right)$$
$$+\int_\Omega \int_0^b \int_0^b \frac{|\phi(\mathbf{r}(x,t),t,s)|^2 |\phi(\mathbf{r}(\tilde{x},t),t,p)|^2 (\mathbf{r}(x,t)-\mathbf{r}(\tilde{x},t))}{|\mathbf{r}(x,t)-\mathbf{r}(\tilde{x},t)|^3}\, d\tilde{x}dsdp$$
$$+\rho \mathbf{g} = \mathbf{0}, \tag{5}$$

in $\Omega \times [0, t_f]$, so that

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \frac{\partial(\rho \mathbf{v})}{\partial X_j}\frac{\partial X_j}{\partial t}$$
$$+\frac{d\nabla(\rho \hat{P})}{dt} - \frac{d(\nabla(\lambda_1 P))}{dt} - \frac{d(\lambda_1 \rho \mathbf{g})}{dt}$$
$$+\frac{d}{dt}\left(\int_0^b |\phi(\mathbf{r}(x,t),t,s)|^2 \hat{v}\, ds\right)$$
$$+\int_\Omega \int_0^b \int_0^b \frac{|\phi(\mathbf{r}(x,t),t,s)|^2 |\phi(\mathbf{r}(\tilde{x},t),t,p)|^2 (\mathbf{r}(x,t)-\mathbf{r}(\tilde{x},t))}{|\mathbf{r}(x,t)-\mathbf{r}(\tilde{x},t)|^3} d\tilde{x}dsdp$$
$$+\rho \mathbf{g} = \mathbf{0}, \tag{6}$$

in $\Omega \times [0, t_f]$, that is,

$$\begin{aligned}
&\frac{\partial(\rho \mathbf{v})}{\partial t} + \frac{\partial(\rho \mathbf{v})}{\partial X_j} v_j \\
&+\nabla P - \frac{d(\nabla(\lambda_1 P))}{dt} - \frac{d(\lambda_1 \rho \mathbf{g})}{dt} \\
&+\frac{d}{dt}\left(\int_0^b |\phi(\mathbf{r}(x,t),t,s)|^2 \hat{v}\, ds\right) \\
&+\int_\Omega \int_0^b \int_0^b \frac{|\phi(\mathbf{r}(x,t),t,s)|^2 |\phi(\mathbf{r}(\tilde{x},t),t,p)|^2 (\mathbf{r}(x,t) - \mathbf{r}(\tilde{x},t))}{|\mathbf{r}(x,t) - \mathbf{r}(\tilde{x},t)|^3} d\tilde{x} ds dp \\
&+\rho \mathbf{g} = \mathbf{0}, \qquad (7)
\end{aligned}$$

in $\Omega \times [0, t_f]$,

To clarify the notation, we highlight that denoting

$$\rho \hat{P} = \rho(\mathbf{r}(x,t),t)\hat{P}(x,t) \equiv W(x,t),$$

we have

$$\frac{\partial(\nabla(\rho \hat{P}))}{\partial t} = \frac{\partial \nabla(W(x,t))}{\partial t} = \nabla\left(\frac{\partial W(x,t)}{\partial t}\right).$$

The variation of J in λ stands for

$$\mathbf{v} - \frac{\partial \mathbf{r}}{\partial t} = \mathbf{0}.$$

The variation of J in $\hat{P}$ stands for

$$\frac{d\rho}{dt} + \rho \frac{\partial}{\partial x_j}\left(\frac{\partial X_j(x,t)}{\partial t}\right) + \frac{d(\lambda_1 \operatorname{div} \mathbf{v})}{dt}\rho = 0, \text{ in } \Omega \times [0, t_f].$$

Denoting $v_j = \frac{\partial X_j(x,t)}{\partial t}$ this last equation stands for

$$\frac{d\rho}{dt} + \rho \frac{\partial(v_j)}{\partial x_j} + \frac{d(\lambda_1 \operatorname{div} \mathbf{v})}{dt}\rho = 0, \text{ in } \Omega \times [0, t_f].$$

Finally, the variation of J in ρ stands for

$$\frac{1}{2}\mathbf{v} \cdot \mathbf{v} + \frac{\partial \hat{P}}{\partial t} - \hat{P}\frac{\partial v_j}{\partial x_j} - \mathbf{g} \cdot \mathbf{r} - \lambda_1 C_v \frac{dT}{dt} - \lambda_1 \mathbf{g} \cdot \mathbf{v} + \hat{P}\frac{d(\lambda_1 \operatorname{div} \mathbf{v})}{dt} + \lambda_3 = 0,$$

in $\Omega \times [0, t_f]$.

Remark 3.2. *Recalling the pressure field is expressed by*

$$P = \frac{\partial(\rho \hat{P})}{\partial t}$$

we have obtained

$$\begin{aligned} & \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + \frac{P}{\rho} - \frac{1}{\rho}\frac{d\rho}{dt}\hat{P} - \hat{P}\frac{\partial v_j}{\partial x_j} + gX_3(x,t) \\ & -\lambda_1 C_v \frac{dT}{dt} - \lambda_1 \mathbf{g}\cdot\mathbf{v} - \hat{P}\frac{d(\lambda_1\ div\ \mathbf{v})}{dt} + \lambda_3 \\ = \ & \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + \frac{P}{\rho} + gX_3(x,t) - \lambda_1 C_v \frac{dT}{dt} - \lambda_1 \mathbf{g}\cdot\mathbf{v} + \lambda_3 = 0, \end{aligned} \tag{8}$$

in $\Omega \times [0, t_f]$.

In summary, we have obtained the following generalization of the Bernoulli equation

$$\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + \frac{P}{\rho} + gX_3 - \lambda_1 C_v \frac{dT}{dt} - \lambda_1 \mathbf{g}\cdot\mathbf{v} + \lambda_3 = 0, \ \ in\ \Omega \times [0, t_f].$$

The variation of J_1 in ϕ stands for

$$\begin{aligned} & -\gamma_0 \frac{d^2\phi}{dt^2} - \gamma(s)\nabla^2\phi + \phi\frac{d\hat{\mathbf{r}}}{dt}\cdot\frac{d\hat{\mathbf{r}}}{dt} \\ & -\lambda_2\ \phi\frac{d\hat{\mathbf{r}}}{dt}\cdot\frac{d\hat{\mathbf{r}}}{dt} + 2\phi\frac{d\hat{\mathbf{r}}}{dt}\cdot\frac{d\mathbf{r}}{dt} \\ & + \int_\Omega\int_0^b \frac{|\phi(\mathbf{r}(x,t),t,s)||\phi(\mathbf{r}(\tilde{x},t),t,p)|^2}{|\mathbf{r}(x,t)-\mathbf{r}(\tilde{x},t)|^3} d\tilde{x}dp \\ & -E(t,s)\phi - 2\lambda_3\phi = 0, \ \text{in}\ \Omega\times[0,t_f]\times[0,b]. \end{aligned} \tag{9}$$

The variation of J_1 in $\hat{\mathbf{r}}$ stands for

$$-\frac{d(|\phi|^2\frac{d\hat{\mathbf{r}}}{dt})}{dt} + \frac{d(\lambda_2|\phi|^2\frac{d\hat{\mathbf{r}}}{dt})}{dt} - 2\frac{d(|\phi|^2\mathbf{v})}{dt} = \mathbf{0}$$

in $\Omega \times [0, t_f] \times [0, b]$.

The variation of J_1 in T stands for

$$-C_v\ \frac{d(\lambda_1\rho)}{dt} - K\nabla^2\lambda_1 + C_v\lambda_2 = 0, \ \text{in}\ \Omega\times[0,t_f].$$

The variation of J_1in λ_1 stands for

$$C_v\ \rho\ \frac{d\ T(\mathbf{r}(x,t),t)}{dt} - K\nabla^2 T - \rho\mathbf{g}\cdot\mathbf{v} + P\ \text{div}\ (\mathbf{v}) - \frac{\partial Q}{\partial t} = 0,$$

in $\Omega \times [0, t_f]$.

The variation of J_1 in λ_2 stands for

$$C_v\, T - \frac{1}{2}\int_0^b |\phi(\mathbf{r}(x,t),t,s)|^2 \frac{d\hat{\mathbf{r}}(\mathbf{r}(x,t),t,s)}{dt} \cdot \frac{d\hat{\mathbf{r}}(\mathbf{r}(x,t),t,s)}{dt}\, ds = 0,$$

in $\Omega \times [0, t_f]$.

The variation of J_1 in λ_3 stands for

$$\rho(\mathbf{r}(x,t),t) - \int_0^b |\phi(\mathbf{r}(x,t),t,s)|^2\, ds = 0, \text{ in } \Omega \times [0, t_f].$$

The variation of J_1 in N stands for

$$E(t,s) + E_1(t) = 0, \text{ in } [0, t_f] \times [0, b].$$

The variation of J_1 in E_1, stands for

$$\int_0^b N(s,t)\, ds - m_T = 0, \text{ in } [0, t_f].$$

Finally, the variation of J_1 in E, stands for

$$\int_\Omega |\phi(\mathbf{r}(x,t),t,s|^2\, dx - N(s,t) = 0, \text{ in } [0, t_f] \times [0, b].$$

Remark 3.3. *To clarify the notation, we firstly recall that*

$$\begin{aligned} \frac{dT(\mathbf{r}(x,t),t)}{dt} &= \frac{\partial T}{\partial t} + \frac{\partial T}{\partial X_j}\frac{\partial X_j(x,t)}{\partial t} \\ &= \frac{\partial T}{\partial t} + \frac{\partial T}{\partial X_j} v_j. \end{aligned} \tag{10}$$

Moreover, we also recall that generically for a scalar smooth function $\varphi = \varphi(\mathbf{r}(x,t),t)$, *we have*

$$\frac{\partial \varphi}{\partial x_j} = \frac{\partial \varphi}{\partial X_k}\frac{\partial X_k}{\partial x_j}.$$

Thus, denoting

$$\{A_{jk}\} = \left\{\frac{\partial X_k(x,t)}{\partial x_j}\right\}^{-1},$$

the equation

$$\begin{aligned}
&\frac{\partial(\rho\mathbf{v})}{\partial t}+\frac{\partial(\rho\mathbf{v})}{\partial X_j}v_j \\
&+\nabla P-\frac{d(\nabla(\lambda_1 P))}{dt}-\frac{d(\lambda_1\rho\mathbf{g})}{dt}+\frac{d}{dt}\left(\int_0^b|\phi(\mathbf{r}(x,t),t,s)|^2\hat{v}\right) \\
&+\int_\Omega\int_0^b\int_0^b\frac{|\phi(\mathbf{r}(x,t),t,s)|^2|\phi(\mathbf{r}(\tilde{x},t),t,p)|^2(\mathbf{r}(x,t)-\mathbf{r}(\tilde{x},t))}{|\mathbf{r}(x,t)-\mathbf{r}(\tilde{x},t)|^3}d\tilde{x}dsdp \\
&+\rho\mathbf{g}=\mathbf{0},
\end{aligned}\tag{11}$$

in $\Omega\times[0,t_f]$, *stands for*

$$\begin{aligned}
&\frac{\partial(\rho\mathbf{v})}{\partial t}+A_{jl}\frac{\partial(\rho\mathbf{v})}{\partial x_l}v_j \\
&+\nabla P-\frac{d(\nabla(\lambda_1 P))}{dt}-\frac{d(\lambda_1\rho\mathbf{g})}{dt} \\
&+\frac{d}{dt}\left(\int_0^b|\phi(\mathbf{r}(x,t),t,s)|^2\hat{v}ds\right) \\
&+\int_\Omega\int_0^b\int_0^b\frac{|\phi(\mathbf{r}(x,t),t,s)|^2|\phi(\mathbf{r}(\tilde{x},t),t,p)|^2(\mathbf{r}(x,t)-\mathbf{r}(\tilde{x},t))}{|\mathbf{r}(x,t)-\mathbf{r}(\tilde{x},t)|^3}d\tilde{x}dsdp \\
&+\rho\mathbf{g}=\mathbf{0},
\end{aligned}\tag{12}$$

in $\Omega\times[0,t_f]$.

Finally, with the Eulerian identification

$$X_k(x,t)\approx x_k$$

and denoting $x_3=z+c_1$, *we obtain*

$$\{A_{kj}\}\approx I_d$$

where I_d *denotes the identity matrix, so that we obtain the final format for such*

Eulerian equation:

$$\begin{aligned}
&\frac{\partial(\rho\mathbf{v})}{\partial t}+\frac{\partial(\rho\mathbf{v})}{\partial x_j}v_j\\
&+\nabla P-\frac{\partial(\nabla(\lambda_1 P))}{\partial t}-\frac{\partial(\nabla(\lambda_1 P))}{\partial x_j}v_j\\
&-\frac{\partial(\lambda_1\rho\mathbf{g})}{\partial t}-\frac{\partial(\lambda_1\rho\mathbf{g})}{\partial x_j}v_j\\
&+\frac{\partial}{\partial t}\left(\int_0^b\phi(x,t,s)^2\hat{v}ds\right)+\frac{\partial}{\partial x_j}\left(\int_0^b\phi(x,t,s)^2\hat{v}ds\right)v_j\\
&+\int_\Omega\int_0^b\int_0^b\frac{|\phi(x,t,s)|^2|\phi(\tilde{x},t,p)|^2(x-\tilde{x})}{|x-\tilde{x}|^3}d\tilde{x}dsdp\\
&+\rho\mathbf{g}=\mathbf{0},
\end{aligned}\tag{13}$$

in $\Omega\times[0,t_f]$.

In such a context, the variation of J *in* $\hat{P}$ *stands for*

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v_j)}{\partial x_j}+\frac{\partial(\lambda_1\ div\ \mathbf{v})}{\partial t}\rho+\frac{\partial(\lambda_1\ div\ \mathbf{v})}{\partial x_j}v_j\rho=0,\ \ in\ \Omega\times[0,t_f].$$

The variation of J *in* ρ *stands for*

$$\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+\frac{P}{\rho}+gx_3-\lambda_1 C_v\frac{\partial T}{\partial t}-\lambda_1 C_v\frac{\partial T}{\partial x_j}v_j+\lambda_1\mathbf{g}\cdot\mathbf{v}+\lambda_3=0,$$

in $\Omega\times[0,t_f]$.

Finally, for the remaining equations, similar remarks hold.

The objective of this section is complete.

4 A note on a more general relativistic context

In this section we address the modeling of a fluid motion in an Eulerian relativistic context.

Let $\Omega\subset\mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

At first we consider such a fluid in question as a particle system one with macroscopic position field $\mathbf{r}:\Omega\times[0,t_f]\to\mathbb{R}^4$, where

$$\mathbf{r}(x,t)=(c\,t,X_1(x,t),X_2(x,t),X_3(x,t)).$$

Here $t \in [0, t_f]$ denotes time and c denotes the speed of light.

For particles p_j with a state corresponding to the eigenvalue $E_j(t)$, we define the related position field by

$$\mathbf{r}_j(x,t) = \mathbf{r}(x,t) + \hat{\mathbf{r}}_j(\mathbf{r}(x,t)),\ \forall j \in \{1, \cdots, N\},$$

where the fields $\{\hat{\mathbf{r}}_j(\mathbf{r}(x,t))\}$ correspond to internal vibration degrees of freedom, related to internal energy and temperature concepts.

The corresponding densities fields are denoted by $|\phi_j(\mathbf{r}(x,t),t)|^2$ with corresponding relativistic densities

$$\frac{|\phi_j(\mathbf{r}(x,t),t)|^2}{\sqrt{1-\frac{v^2}{c^2}}},$$

where

$$v^2 = \left(\frac{\partial X_1(x,t)}{\partial t}\right)^2 + \left(\frac{\partial X_2(x,t)}{\partial t}\right)^2 + \left(\frac{\partial X_3(x,t)}{\partial t}\right)^2,$$

$$\frac{\partial \mathbf{r}(x,t)}{\partial t} \cdot \frac{\partial \mathbf{r}(x,t)}{\partial t} = -c^2 + v^2.$$

Here we have generically denoted

$$\mathbf{y} \cdot \mathbf{z} = -y_0 z_0 + y_1 z_1 + y_2 z_2 + y_3 z_3,$$

$\forall \mathbf{y} = (y_0, y_1, y_2, y_3),\ \mathbf{z} = (z_0, z_1, z_2, z_3) \in \mathbb{R}^4$.

The macroscopic density field at a rest is defined by

$$\rho_1(\mathbf{r}(x,t),t) = \sum_{j=1}^{N} |\phi_j(\mathbf{r}(x,t),t)|^2 \equiv |\phi|^2,$$

and the related relativistic density field is defined by

$$\rho(\mathbf{r}(x,t),t) = \sum_{j=1}^{N} \frac{|\phi_j(\mathbf{r}(x,t),t)|^2}{\sqrt{1-\frac{v^2}{c^2}}},$$

We define also a macroscopic system velocity field by

$$\mathbf{v}(\mathbf{r}(x,t),t)$$

which stands for the system velocity field at the position $\mathbf{r}(x,t)$ and time t.

Moreover, we set the following obvious constraint

$$\mathbf{v}(\mathbf{r}(x,t),t) = \frac{\partial \mathbf{r}(x,t)}{\partial t},\ \text{in } \Omega \times [0, t_f].$$

Finally, the non-constant scalar field of temperature is denoted by

$$T = T(\mathbf{r}(x,t),t),$$

where we define

$$C_v\, T(\mathbf{r}(x,t),t) = \frac{1}{2}\sum_{j=1}^{N} \frac{|\phi_j(\mathbf{r}(x,t),t)|^2}{\sqrt{1-\frac{v^2}{c^2}}} \frac{d\hat{\mathbf{r}}_j(\mathbf{r}(x,t),t)}{dt} \cdot \frac{d\hat{\mathbf{r}}_j(\mathbf{r}(x,t),t)}{dt}.$$

Furthermore, following the approach in [16], we assume the following energy equation

$$C_v\rho \frac{1}{\sqrt{-g}} \frac{d(\sqrt{-g}\, T(\mathbf{r}(x,t),t))}{dt} - K\nabla^2 T - \rho\mathbf{g}\cdot\mathbf{v} + P \text{ div } (\mathbf{v}) - \frac{\partial Q}{\partial t} = 0,$$

in $\Omega \times [0,t_f]$, where as previously indicated,

$$T = T(\mathbf{r}(x,t),t)$$

is the temperature scalar field, $C_v > 0$, $K > 0$ are real constants and Q is a heat function.

Here, concerning the manifold in question, we have denoted

$$\nabla^2 T = \frac{1}{\sqrt{-g}}\left[\frac{\partial}{\partial x_i}\left(g^{ij}\frac{\partial T}{\partial x_j}\sqrt{-g}\right)\right],$$

where $x_0 = t$,

$$\mathbf{g}_k = \frac{\partial \mathbf{r}(x,t)}{\partial x_k},$$

$$g_{jk} = \mathbf{g}_j \cdot \mathbf{g}_k,\ \forall j,k \in \{0,1,2,3\},$$

$$\{g^{jk}\} = \{g_{jk}\}^{-1}$$

and

$$g = \det\{g_{jk}\}.$$

Moreover, also considering the manifold in question, we have denoted

$$\text{div } \mathbf{v} = \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g}\, v_j)}{\partial x_j}.$$

Finally, we assume the following mass conservation equation as a constraint

$$\frac{1}{\sqrt{-g}} \frac{d(\sqrt{-g}\,\rho)}{dt} + \rho \frac{1}{\sqrt{-g}} \frac{\partial(v_j\sqrt{-g})}{\partial x_j} = 0, \text{ in } \Omega \times [0,t_f].$$

Concerning such a fluid motion modeling, neglecting the self interaction term and denoting generically $Y = C^1$, $Y_1 = C^2$ and $V = C_0^2$, we define the following variational formulation represented by the functional $J : Y \times V \times Y_1 \to \mathbb{R}$ already including the concerning Lagrange multipliers for the constraints, where

$$
\begin{aligned}
& J(\mathbf{v},\mathbf{r},\hat{\mathbf{r}},\{\phi_j\},\rho,T,\lambda,\hat{P},E,\mu,N) \\
= \; & \frac{1}{2}\int_0^{t_f}\int_\Omega \rho\mathbf{v}(\mathbf{r}(x,t),t)\cdot\mathbf{v}(\mathbf{r}(x,t),t)\,\sqrt{-g}\,dx\,dt \\
& +\frac{1}{2}\sum_{j=1}^{N}\int_0^{t_f}\int_\Omega \frac{|\phi_j(\mathbf{r}(x,t),t)|^2}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\hat{\mathbf{r}}_j(\mathbf{r}(x,t),t)}{dt}\cdot\frac{d\hat{\mathbf{r}}_j(\mathbf{r}(x,t),t)}{dt}\,\sqrt{-g}\,dxdt \\
& +\sum_{j=1}^{N}\int_0^{t_f}\int_\Omega \frac{|\phi_j(\mathbf{r}(x,t),t)|^2}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\hat{\mathbf{r}}_j(\mathbf{r}(x,t),t)}{dt}\cdot\frac{d\mathbf{r}(x,t)}{dt}\,\sqrt{-g}\,dxdt \\
& +\left\langle \lambda, \mathbf{v}(\mathbf{r}(x,t),t)-\frac{\partial\mathbf{r}(x,t)}{\partial t}\right\rangle_{L^2} \\
& -\int_0^{t_f}\int_\Omega \hat{P}\left(\frac{1}{\sqrt{-g}}\frac{d(\sqrt{-g}\rho_1)}{dt}+\frac{\rho_1}{\sqrt{-g}}\frac{\partial(v_j\sqrt{-g})}{\partial x_j}\right)\sqrt{-g}\,dxdt \\
& -\int_0^{t_f}\int_\Omega \rho\mathbf{g}\cdot\mathbf{r}(x,t)\sqrt{-g}\,dxdt \\
& +\int_0^{t_f}\int_\Omega \lambda_1\left(C_v\rho\frac{1}{\sqrt{-g}}\frac{d(\sqrt{-g}\,T(\mathbf{r}(x,t),t))}{dt}-K\nabla^2T-\rho\mathbf{g}\cdot\mathbf{v}\right. \\
& \left.+P\ \mathrm{div}\ (\mathbf{v})-\frac{\partial Q}{\partial t}\right)\sqrt{-g}\,dxdt \\
& +\int_0^{t_f}\int_\Omega \lambda_2\left(C_v\,T-\frac{1}{2}\sum_{j=1}^{N}\frac{|\phi_j|^2}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\hat{\mathbf{r}}_j(\mathbf{r}(x,t),t)}{dt}\cdot\frac{d\hat{\mathbf{r}}_j(\mathbf{r}(x,t),t)}{dt}\right)\sqrt{-g}\,dxdt \\
& +\frac{\gamma}{2}\int_0^{t_f}\int_\Omega R\sqrt{-g}\,dxdt \\
& -\sum_{j=1}^{N}\int_0^{t_f}E_j(t)\left(\int_\Omega|\phi_j(\mathbf{r}(x,t),t)|^2\,\sqrt{-g}\,dx-N_j(t)m_p\right)dt \\
& -\int_0^{t_f}\mu(t)\left(N_0-\sum_{j=1}^{N}N_j(t)\right)dt \\
& +\sum_{j=1}^{N}\int_0^{t_f}\int_\Omega T\,\frac{N_j(t)}{N_0}\ln\left(\frac{N_j(t)}{N_0\,e}\right)\sqrt{-g}\,dxdt \\
& +\int_\Omega\lambda_3\left(\rho(\mathbf{r}(x,t),t)-\sum_{j=1}^{N}\frac{|\phi_j(\mathbf{r}(x,t),t)|^2}{\sqrt{1-\frac{v^2}{c^2}}}\right)\sqrt{-g}\,dxdt. \qquad (14)
\end{aligned}
$$

$$R = g^{jk} R_{jk},$$
$$R_{jk} = R_e[R^i_{jik}(\phi, g)],$$

and

$$\begin{aligned} R^l_{ijk}(\phi, g) &= |\phi|^2 \hat{R}^l_{ijk} \\ &\quad + \frac{\partial \phi}{\partial x_i} \frac{\partial \phi^*}{\partial x_j} \delta_{kl} \\ &\quad + \frac{\partial \phi}{\partial x_j} \phi^* \Gamma^l_{ik}. \end{aligned} \tag{15}$$

Finally, the Riemann curvature tensor stands for

$$\hat{R}^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x_i} - \frac{\partial \Gamma^l_{ik}}{\partial x_j} + \Gamma^p_{jk} \Gamma^l_{ip} - \Gamma^p_{ik} \Gamma^l_{jp}. \tag{16}$$

and for the Levi-Civita connection,

$$\Gamma^i_{jk} = g^{il} \left(\frac{\partial g_{kl}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_k} - \frac{\partial g_{jk}}{\partial x_l} \right), \ \forall i, j, k, l \in \{0, 1, 2, 3\}.$$

where the pressure field P is defined by

$$P = P(\hat{P}, \rho) = \frac{d(\hat{P}\rho)}{dt},$$
$$\gamma_j > 0, \ \forall j \in \{1, \cdots, N\}$$

and $\gamma > 0$ are appropriate real constants.

Finally, the entropy differential is defined by

$$dS(x,t) = -\sum_{j=1}^{N} \frac{N_j(t)}{N_0} \ln \left(\frac{N_j(t)}{N_0 \, e} \right) \sqrt{-g} \, dxdt,$$

so that

$$\int_0^{t_f} \int_\Omega T \, dS = -\sum_{j=1}^{N} \int_0^{t_f} \int_\Omega T(\mathbf{r}(x,t), t) \frac{N_j(t)}{N_0} \ln \left(\frac{N_j(t)}{N_0 \, e} \right) \sqrt{-g} \, dxdt.$$

Here N_0 denotes a fixed total number of particles and $N_j(t)$ the number of particles at the state corresponding to $E_j(t)$, $\forall j \in \{1, \cdots, N\}$, so that

$$N_0 = \sum_{j=1}^{N} N_j(t), \ \text{in} \ [0, t_f].$$

Remark 4.1. *Observe that the variation of J in N_j give us*

$$-E_j(t)m_p - \mu(t) - \int_\Omega T\sqrt{-g}dx\frac{\partial}{\partial N_j(t)}\left(\frac{N_j(t)}{N_0}\ln\left(\frac{N_j(t)}{N_0}\right) - \frac{N_j(t)}{N_0}\right),$$

so that

$$-E_j(t)m_p - \mu(t) - \int_\Omega T\sqrt{-g}\,dx\left(\frac{1}{N_0}\ln\left(\frac{N_j(t)}{N_0}\right) + \frac{1}{N_0} - \frac{1}{N_0}\right),$$

that is,

$$-E_j(t)m_p - \mu(t) - \int_\Omega T\sqrt{-g}\,dx\frac{1}{N_0}\ln\left(\frac{N_j(t)}{N_0}\right),$$

and thus

$$\frac{N_j(t)}{N_0} = e^{N_0\left(\frac{-E_j(t)m_p-\mu(t)}{\int_\Omega T\sqrt{-g}\,dx}\right)},\ \textit{in}\ [0,t_f].$$

Such a distribution in $N_j(t)$ obtained is similar, in some sense, to the well known Gibbs one obtained in statistical physics.

5 Conclusion

In this article we have developed variational formulations for modeling a fluid motion in an Eulerian context including a scalar temperature field.

In this second work we have addressed a relativistic case as well. We highlight the number N is in general very large so that tools of statistical physics may be a possibility to address such a high magnitude for N. Anyway, we believe the internal energy variables may be appropriately modeled in such a more general quantum relativistic context.

Finally, for the second non-relativistic model, we emphasize to have obtained an additional term involving the Lagrange multiplier λ_1 in the continuity equation. We believe such a mentioned part results from the effect of the internal energy variables on the mass conservation equation.

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